TABLE I

<table>
<thead>
<tr>
<th>512 FFT</th>
<th>without bit reversal</th>
<th>with bit reversal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cooley-Tukey</td>
<td>0.32ms</td>
<td>0.40ms</td>
</tr>
<tr>
<td>Our Algorithm</td>
<td>0.35ms</td>
<td></td>
</tr>
</tbody>
</table>

if we have \( Q(l, k) \) and \( X_m \) in each stage to operate on independent indices

\[
(Q(l, k)X_m)(n_{K_1}, \ldots, n_{k}, \ldots, n_m, \ldots, n_{i_1}, \ldots, n_l)
\]

\[
= y(n_{K_1}, \ldots, n_{k}, \ldots, n_m, \ldots, n_{i_1}, \ldots, n_l).
\]

Set \( N = p^K \), \( K \) = odd, we have

\[
F(N) = X^K((Q(1, 1)X^K_{K-1})(Q(K - 1, 2)X^K_{K-2}) \cdots \frac{(Q((K + 1)/2 + 1, (K - 1)/2)X^K_{K+1/2})}{X^K_{K-1/2}} \cdots X^K_{1}).
\]

(12)

Set \( N = p^K \), \( K \) = even, we have

\[
F(N) = X^K((Q(1, 1)X^K_{K-1})(Q(K - 1, 2)X^K_{K-2}) \cdots \frac{(Q((K + 1)/2 + 1, (K - 1)/2)X^K_{K+1/2})}{X^K_{K-1/2}} \cdots X^K_{1}).
\]

(13)

Taking the transpose of (12) and (13), we will have the decimation-in-time version of the self-sorting in-place FFT algorithm. It is seen that, in our algorithm, the \( Q(l, k) \)'s and \( X_m \)'s act on independent indices at every stage. It is conceivable that in the stage with operator \( Q(l, k) \) and \( X_m \), we only need to swap indices \( l \) and \( k \) while \( X_m \) acts on index \( m \). Since \( l, k, m \) are independent, swapping can be done with one pair of \( p \)-point data at a time. So a working space of length \( 2p \) is enough for the above stage. In practice, we only need \( p + 1 \) temporary data space to sort the output data. The strategy is to update one point at a time rather than the whole group.

To test our algorithm, we implemented it on a typical RISC processor, the Intel i860. Both the Cooley-Tukey radix-8 FFT and our self-sorting, in-place FFT algorithms are programmed in assembly language to get optimum performance. The timing results are given in Table I for comparison. Temperton's algorithm will not be practical in this case since there are not enough temporary registers for the self-sorting stage.

There is a slight difference of CPU time between the Cooley-Tukey algorithm without scrambling and our self-sorting, in-place algorithm. This is due to the fact that self-sorting in-place program is longer and does not fit the instruction cache memory (4 kbytes) of the i860. Still we see that the new algorithm is about 14% faster than the traditional Cooley-Tukey algorithm.

Decomposition of the Wigner-Ville Distribution and Time-Frequency Distribution Series
Shie Qian and Dapang Chen

Abstract—Using the orthogonal-like Gabor expansion, we decompose the Wigner-Ville distribution (WVD) to a linear combination of localized and symmetric functions \( \text{WVD}_{m,n}(t, \omega) \), the WVD of the Gabor elementary functions, \( h_{m,n}(t) \) and \( h_{m,n}^*(t) \). Since the influence of the \( \text{WVD}_{m,n}(t, \omega) \) to the useful properties is inversely proportional to the distance between \( h_{m,n}(t) \) and \( h_{m,n}^*(t) \), the \( \text{WVD}_{m,n}(t, \omega) \) are further grouped as a series of the function \( P_2(t, \omega) \). We name the resulting representation the time-frequency distribution series (TFDS) (also known as the Gabor Spectrogram in industry). The TFDS \( 0 \) consists of up to a \( D \)-th order \( P_2(t, \omega) \). While TFDS \( 0 \) is similar to the spectrogram, TFDS \( 0 \) converges to the WVD \( (t, \omega) \). Numerical simulations demonstrate that adjusting the order \( D \) of the TFDS, one could effectively balance the cross-term interference and the useful properties.

I. INTRODUCTION

The Wigner-Ville distribution (WVD) has received great attention in the signal processing community for many years. For signals, \( s_1(t) \) and \( s_2(t) \), the WVD is defined as [9], [10]

\[
\text{WVD}_{s_1 s_2}(t, \omega) = \int s_1(t + \frac{\tau}{2})s_2^*(t - \frac{\tau}{2}) \exp(-j\omega\tau) \, d\tau. \quad (1)
\]

Although the potential of the Wigner-Ville distribution for signal processing has long been recognized [2], [4], its applications are limited mainly due to a cross-term interference problem.

One way to identify the auto- and cross-terms is to take the Fourier transform with respect to the "instantaneous auto-correlation" function, \( s(t + \frac{1}{2})s^*(t - \frac{1}{2}) \), and thereby obtain the ambiguity function.
It can be shown that the auto-terms are always connected to the origin \((0,0)\) and the cross-terms tend to spread to other places. This observation suggests that one may apply some weighting functions (or kernel functions) \(\phi(\theta, \tau)\) to suppress \(A_\theta(\theta, \tau)\) for \(\theta, \tau \gg 0\) and thereby reduce the cross-term interference. The resulting representations are usually called the Cohen's class [3]. For example

\[
P(t, \omega) = \frac{1}{2\pi} \int \int R_\theta(\theta, \tau) \exp(-j[\omega \tau + \theta t]) \, d\theta d\tau
\]

where

\[
R_\theta(\theta, \tau) = A_\theta(\theta, \tau) \phi(\theta, \tau).
\]

Figs. 1 and 2 depict FSK and PSK signals and their corresponding \(A_\theta(\theta, \tau)\) in the half \(-\pi < \theta < \pi, 0 \leq \tau < \infty\) plane. The \(A_\theta(\theta, \tau)\) corresponding to auto-terms do connect to the origin (0,0), but the entire shapes of the auto-terms of \(A_\theta(\theta, \tau)\) could be so different that it is hard to design \(\phi(\theta, \tau)\) able to retain the auto-terms and effectively suppress the cross-terms in all applications.

Besides being identified in the "ambiguity plane", the cross-terms can also be characterized in the "Wigner-Ville plane" if the auto-terms are known. For example, any pair of signals creates one cross-term in their midpoint. Based on this fact, we decompose the signal \(s(t)\), via the orthogonal-like Gabor expansion, into time- and frequency-shifted Gaussian functions [7], [8]. Then, we compute the Wigner-Ville distribution with respect to the orthogonal-like Gabor expansion. Finally, based on the contributions to those useful properties, we regroup the WVD of elementary functions as the series of time-frequency functions \(P_d(t, \omega)\). The resulting representation is named the time-frequency distribution series (TFDS) (also known as the Gabor Spectrogram in industry). TFDS\(_D\) consists of up to \(D\)th order time-frequency functions \(P_d(t, \omega)\). While TFDS\(_0\) = \(P_0(t, \omega)\) is nonnegative and has less cross-term interference (similar to the spectrogram using a Gaussian window function), TFDS\(_\infty\) converges to the WVD. One can easily balance the cross-term interference and the useful properties simply by adjusting the order \(D\) of the TFDS.
Moreover, if \( s(t) = h(t) + h'(t) \), then

\[
\text{WVD}_d(t, \omega) = \text{WVD}_h(t, \omega) + \text{WVD}'_h(t, \omega) + 2\text{Re}\{\text{WVD}_{h,h'}(t, \omega)\}
\]  

(3)

where \( \text{WVD}_{h,h'}(t, \omega) \) is

\[
\text{WVD}_{h,h'}(t, \omega) = \exp\{j \omega_t \sigma_t \} H(t - t_\mu, \omega - \omega_\mu)
\]

where

\[
H(t, \omega) = 2 \exp \left\{-\left[\frac{t^2}{\sigma^2} + (\sigma \omega)^2\right]\right\} \exp\{-j[\omega_t \sigma_t + \omega_\mu \sigma_\mu]\}
\]

\[
t_\mu = \frac{m + m'}{2} T, \quad \omega_\mu = \frac{n + n'}{2} \Omega.
\]

The \( \text{WVD}_{h,h'}(t, \omega) \) has the same envelope as the \( \text{WVD}_h(t, \omega) \), but is oscillating with \( \omega_\mu \) in the time domain and \( \omega_\mu \) in the frequency domain, respectively (for \( t_\mu = \omega_\mu = 0 \), the \( H(t, \omega) \) reduces to the \( \text{WVD}_h(t, \omega) \)). The location of \( \text{WVD}_{h,h'}(t, \omega) \) is halfway between \( h \) and \( h' \). The \( 2\text{Re}\{\text{WVD}_{h,h'}(t, \omega)\} \) is the so-called cross-term. Although the cross-term does not have a physical interpretation, it has twice the magnitude of the auto-term.

Equation (3) demonstrates a general way to distinguish the cross-terms based on the knowledge of the auto-terms. In fact, any pair of signals creates one cross-term at their midpoint [6]. This suggests that if the signal \( s(t) \) can be decomposed as a linear combination of some elementary functions \( h(t) \) then we may be able to better control the cross-terms.

As we know, the signal \( s(t) \) can be decomposed in many ways. Intuitively, for desired performance the decomposition has to be such that:

1) the WVD of the elementary functions \( h \) should be as concentrated as possible;
2) the weight of the elementary WVD centered at \( (t_\mu, \omega_\mu) \) has to reflect the signal behavior in the vicinity of \( (t_\mu, \omega_\mu) \).

Apparently, the orthogonal-like Gabor expansion [7] is one ideal candidate. Because

1) the elementary functions used in the Gabor expansion [5] are time- and frequency-shifted Gaussian functions. The WVD of the elementary functions are not only optimally concentrated in the joint time-frequency domain, but also can be obtained simply by time- and frequency-shifting the single prototype function \( H(t, \omega) \).
2) The coefficients of the orthogonal-like Gabor expansion are projections of the signal \( s(t) \) on the individual elementary Gaussian functions. Therefore, the weight of the WVD of elementary functions describe the signal’s local behavior.

For a signal \( s(t) \), the orthogonal-like Gabor expansion is defined as [5]

\[
s(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{m,n} g(t - mT) \exp\{jn\Omega t\}
\]

(5)

where \( T \) and \( \Omega \) are the time and frequency sampling steps, respectively. The Gabor coefficients \( C_{m,n} \) are computed by a regular inner product rule, i.e.

\[
C_{m,n} = \frac{1}{\sqrt{2\pi}} \int s(t) g^*(t - mT) \exp\{-jn\Omega t\} dt
\]

\[
\approx \alpha \int s(t) g^*(t - mT) \exp\{-jn\Omega t\} dt
\]

(6)

For the presentation simplicity, we omit \( m, m', n, n' \) from \( H(t, \omega), t_\mu, \omega_\mu, t_\mu, \omega_\mu \), although they all are functions of \( m, n, m', \) and \( n' \).
Fig. 5. Comparison of WVD, Choi-Williams distribution, spectrogram, and TFDS. (a) \( s(t) \).
(b) \( R(t, \tau) = s(t + \tau/2)s^*(t - \tau/2) \). (c) \(|A(\theta, \tau)| = \int R(t, \tau)\exp(-j\omega \tau)dt| \). (d) \( \text{WVD}(t, \omega) = \int R(t, \tau)\exp(-j\omega \tau)dt \). (e) \( \phi(\theta, \tau) = \exp(-\alpha \theta^2 \tau^2) \).

where \( \alpha \) is a real-valued constant and the \( \gamma_{opt}(t) \) is the biorthogonal auxiliary function that is most similar to the \( g(t) \) \cite{7}. To ensure \( \gamma_{opt}(t) \approx \omega g(t) \), in general, an oversampling scheme has to be employed. As indicated in \cite{7}, at quadruple oversampling, the \( \gamma_{opt}(t) \) is virtually identical to the \( \omega g(t) \) if the time sampling step \( T \), the frequency sampling step \( \Omega \), and the variance \( \sigma^2 \) are properly selected.

As shown in (6), \( C_{m,n} \) is a sampled short-time Fourier transform (STFT). Since the basis function \( g(t - mT)\exp(jn\Omega t) \) is concentrated at \( (mT, n\Omega) \), \( C_{m,n} \) gives a measure of how much of the signal \( s(t) \) is presented in the vicinity of \( (mT, n\Omega) \).

Taking the WVD on both sides of (5) yields

\[
\text{WVD}(t, \omega) = \sum_{m = -\infty}^{\infty} \sum_{n = -\infty}^{\infty} C_{m,n} C_{m',n'}^* \text{WVD}_{h_{m',n'}}(t, \omega).
\]

(7)

The locations of the \( \text{WVD}_{h_{m',n'}}(t, \omega) \) are illustrated in Fig. 3. The right side of (7) and Fig. 3 show that the WVD is composed of localized and symmetric (in both time and frequency) elementary functions. Each elementary function is the superposition of the oscillated time-frequency functions with a 2-D Gaussian envelope

\[
2\{C_{m,n} C_{m',n'}^* \exp{j\omega_0 t_0} \} \times \exp\left\{-\left[\frac{t^2}{\sigma^2} + (\omega - \omega_0)^2\right]\right\} \exp\{-j[t\omega - \omega_0 t]\}
\]

where \( t_0 \) and \( \omega_0 \) reflect the degree of oscillation.

For \( m = m' \) and \( n = n' \),

\[
C_{m,n} C_{m',n'}^* \text{WVD}_{h_{m',n'}}(t, \omega)
\]

\[
= 2|C_{m,n}|^2 \exp\left\{-\left[\frac{(t - mT)^2}{\sigma^2} + |\omega - \omega_0|^2\right]\right\}
\]

which is real and nonnegative. For \( m \neq m' \) or \( n \neq n' \),

\[
C_{m,n} C_{m',n'}^* \text{WVD}_{h_{m',n'}}(t, \omega)
\]

\[
= C_{m,n} C_{m',n'}^* \exp{j\omega_0 t_0} H(t - t_0, \omega - \omega_0)
\]

which is oscillating and centered at \((t_0, \omega_0)\). The degree of the oscillation is proportional to the distance between \( h_{m,n}(t) \) and \( h_{m',n'}(t) \) as described in (4). It is the \( \text{WVD}_{h_{m',n'}}(t, \omega) \) (for \( h \neq h' \)) that is the source of the negative values and possible cross-term interference. However, the overall contribution of the each individual
term, $\text{WVD}_{h,n}(t, \omega)$, to the useful properties decrease in an inverse proportion to the distance between the elementary functions $h_m, n(t)$ and $h_{m', n'}(t)$. For example

$$A(t_d) = 2(2\pi \sigma^2)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}(t - t_d)^2\right\} \times \exp\{i(\omega_d t + \omega_d t_d)\}$$

where

$$A(t_d) = 2(\pi \sigma^{-2})^{1/2} \exp\{-\frac{1}{2\sigma}t_d^2\}.$$  

The right side of (8) is the sum of the dc component ($\omega_d = 0$) and a group of the harmonics ($\omega_d \neq 0$). The magnitude of the harmonic terms, $A(t_d)$, is inversely proportional to $t_d$, the distance between the two elementary functions in the time domain. The frequency $\omega_d$ is equal to the distance of the two elementary functions in the frequency domain. In other words, the farther apart the pair of elementary functions $h_m, n(t)$ and $h_{m', n'}(t)$ are, the less the energy and the higher harmonics the $\text{WVD}_{h,n}(t, \omega)$ contain. It is easier to verify that similar observations also hold for all other properties. Therefore, one can selectively remove those $\text{WVD}_{h,n}(t, \omega)$ that are caused by well-separated elementary functions, $h_{m, n}(t)$ and $h_{m', n'}(t)$, to balance the cross-term interference and useful properties.

III. TIME-FREQUENCY DISTRIBUTION SERIES

Based upon the analysis presented in the previous section, we develop the time-frequency distribution series (TFDS) as

$$\text{TFDS}_D(t, \omega) \triangleq \sum_{d=0}^D P_d(t, \omega)$$

where $D$ is the order of the TFDS. $P_d(t, \omega)$ are 2-D linear interpolation filters defined as

$$P_d(t, \omega) \triangleq 2 \sum_{\Lambda} a(t, \omega) H_{\Lambda}(t - t_d, \omega - \omega_d)$$

where $\Lambda$ denotes the set of pairs of coordinates, $(m, n)$ and $(m', n')$, which are separated in $d$, i.e.

$$\Lambda = \{(m, m'), (n, n') | m - m' = n - n' = d\}$$

where $C_{m,n}$ is the orthogonal-like Gabor expansion coefficient computed via (6).

For $d = 0$, the $P_0(t, \omega)$ is a 2-D linear time-frequency invariant interpolation filter, i.e.

$$P_0(t, \omega) = 2 \sum_{m,n} |C_{m,n}|^2 \exp\{-\frac{1}{2\sigma^2}(t - mT)^2 + i(\omega - n\Omega)^2\}.$$ 

The filter input $|C_{m,n}|^2$ is a sampled spectrogram using a Gaussian window function. The filter impulse response is a 2-D localized Gaussian function, which is low-pass and time-frequency invariant. Hence, $\text{TFDS}_0(t, \omega) = P_0(t, \omega)$ is similar to the spectrogram describing the trajectory of the signal energy distribution in the joint time-frequency domain. As $D$ goes to infinity, the $\text{TFDS}_\infty$ converges to the WVD.

The following numerical simulations are used to demonstrate the effectiveness of the TFDS. In all examples, the elementary function
Fig. 6. (a) Wigner-Ville distribution. (b) Cone-shaped distribution. (c) Choi-Williams distribution. (d) Spectrogram. (e) TFD&(t, Uj).

While roughly possessing the same resolution as the WVD(t, Uj), the TFD&(t, Uj) basically eliminates the cross-terms appearing in the Wigner-Ville distribution.

\[ g(k) = (\pi \sigma^2)^{-0.25} \exp \left\{ -\frac{k^2}{2\sigma^2} \right\}, \quad \text{for} \quad -128 \leq k < 127 \]

where \( \sigma^2 = 162.98 \). The discrete-time sampling step is 16. The discrete frequency sampling step is 4. The oversampling rate is four. In this case, as shown in [7], the difference between the \( \gamma \text{opt}(k) \) and \( g(k) \) is negligible, which ensures that the Gabor coefficients \( C_{m,n} \) computed by (6) reflect the local behavior of the signal.

**(Fig. 7. (a) TFDS0(t, u). (b) TFDS1(t, u). (c) TFDS2(t, u). (d) P(t, u).)**

The resolution is improved as the order increases. On the right hand, the cross-terms become more prominent. Fig. 7(d) illustrates the higher order term \( P(t, u) \) has significant contribution to the cross-term, though it contains less signal energy.

Fig. 4 depicts the TFDS of a PSK signal. It can be shown that the TFDS0(t, u) is very similar to a spectrogram using a Gaussian window, which is nonnegative and has minimal cross-term interference, however, the resolution of TFDS0(t, u) is rather poor. As \( D \) gets larger, the resolution gets better, but the cross-terms become more prominent. Finally, the TFDS converges to the WVD. The best compromise between resolution and cross-term interference has been found about \( D = 3 \sim 4 \).

The signal \( s(t) \) in Fig. 5(a) contains four Gaussian envelope functions with different time and frequency centers. Combining the time waveform and the "instantaneous autocorrelation" function (Fig. 5(b)), we can point out the auto- and cross-term in the "ambiguity plane" (Fig. 5(c)). Except for the cluster concentrated in the middle of the bottom line, all other clusters correspond to the cross-terms. Fig. 5(e) depicts the weighting function (or kernel...
function $\alpha(\theta, \tau)$ used for the Choi-Williams distribution (CWD). As shown in Fig. 5(e), in the CWD case the $|A_\nu(\theta, \tau)|\alpha(\theta, \tau)$ not only alters the autotermals but also contains significant cross-terms (all those cross-terms that are close to the $x$-axis ($\theta$) or the $y$-axis ($\tau$) are preserved). Fig. 5(d), 5(f), 5(h), and 5(i) describe the WVD, the CWD, spectrogram, and TFDS, respectively. Obviously, the TFDS yields the best result. Figs. 6 and 7 are corresponding 3-D plots.

IV. CONCLUSION

Based on the decomposition of the Wigner-Ville distribution, we represent the WVD in terms of time-frequency functions $P_d(t, \omega)$ and thereby define the time-frequency distribution series (TFDS). The index $d$ indicates the degree of oscillation of the function $P_d(t, \omega)$. While the $P_d(t, \omega)$ is nonnegative and plays the most important role in all the useful properties, the $P_{\infty}(t, \omega)$ exhibits the highest oscillation and possesses the least useful properties. Therefore, the TFDS, consisting of up to a Dth order $P_d(t, \omega)$, provides a feasible way to balance the cross-term interference and those useful properties. As the order $D$ gets larger the resolution becomes better, however, cross-interference becomes more prominent. The best compromise occurs when $D \approx 3 \sim 4$. The lower order TFDS not only preserves desirable properties but is also computationally economic.

The ideas presented in this paper are simple and effective, but could not be realized without the recent breakthroughs in the Gabor expansion implementation. Over the past two years, the TFDS has been tested in a variety of areas, such as nondestructive evaluations, aneurysm research, speech analysis, and economic data analysis. The TFDS seems more effective and robust, in terms of cross-term suppression, compared to other techniques.

ACKNOWLEDGMENT

The authors wish to express their appreciation to their colleague, Crystal Doubrava, for her careful reading of the manuscript and many valuable suggestions. The authors would also like to thank referees for providing constructive comments.

REFERENCES


Abstract—A technique using Jacobian elliptic functions is given that, by removing a previous method's double-zero constraint, yields improved designs of linear phase IIR filters.

In theory, it is possible to implement linear phase IIR filters as a tandem connection of an arbitrary transfer function $H(z)$ and a time-reversed version of the same function $H(z^{-1})$. Powell and Chau have devised a clever technique for doing this by approximating a local (in time) time-reversal operation using two copies of a desired IIR filter, several blocks of storage registers, and control circuitry that accesses two of these register blocks on a "last-in, first-out" (LIFO) basis. There is, however, a certain inefficiency in the use of identical transfer functions in the $H(z^{-1})H(z)$ cascade because the stopband transmission zeros all appear as "double zeros" in any such system.

We have devised a design technique, using Jacobian elliptic functions, for an optimal pair of transfer functions that, when cascaded as $H_1(z^{-1})H_2(z)$ in the same type of IIR time-reversal system, will meet the same linear-phase design specifications as that of [1] but also yields an additional 6 dB of stopband loss. This extra loss can be traded for lower passband ripple and/or a narrower transition band by a simple revision of the design specifications.

The technique of [1], of course, yields only an approximate linear-phase design because it happens that certain errors are inevitable in the implementation of the time-reversal process due to the finite length of the register blocks and the infinite length of the filter's impulse response. For large enough register blocks, the approximation can be quite acceptable, but this required length grows as the filter specifications become more demanding. When the register-block length is not quite sufficient, errors in the passband magnitude and phase response occur. Our investigations indicate that these effects are less pronounced in our $H_1(z^{-1})H_2(z)$ design than they are in the design of [1]; thus, this represents another advantage for our modified approach.

To illustrate our design technique, consider the Fig. 1(a) pole-zero plot of a lowpass filter $H(z)$. The corresponding pole-zero pattern of $H(z^{-1})$ is shown in Fig. 1(b), and the overall filter designed as in [1] would have the pole-zero pattern shown in Fig. 1(c). In principle, a transfer function having the same set of Fig. 1(c) poles, but whose zeros (while confined to the unit circle) were not constrained to have order two, would be better able to distribute the zeros throughout the filter's stopband, thereby yielding a more efficient transfer function.

Manuscript received May 27, 1993; revised February 6, 1994. This work was supported by the National Science Foundation under Grant MIP-9201104 and by the Office of Naval Research under Grant N00014-91-1-1852. The associate editor coordinating the review of this paper and approving it for publication was Prof. Tamal Bose. The authors are with the Department of Electrical Engineering, University of California, Los Angeles 90024-1594 USA. IEEE Log Number 9403754.