Numerical Methods in Economics

MIT Press, 1998

Notes for Chapter 4
Optimization

Kenneth L. Judd
Hoover Institution

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Optimization Problems

- Canonical problem:

$$\begin{aligned} \min_x & \, f(x) \\ s.t. \; & g(x) = 0, \\ & h(x) \leq 0, \end{aligned}$$

  - $f : \mathbb{R}^n \to \mathbb{R}$ is the *objective function*
  - $g : \mathbb{R}^n \to \mathbb{R}^m$ is the vector of m *equality constraints*
  - $h : \mathbb{R}^n \to \mathbb{R}^\ell$ is the vector of ℓ *inequality constraints*.

- Examples:

  - maximization of consumer utility subject to a budget constraint
  - optimal incentive contracts
  - portfolio optimization
  - life-cycle consumption

- Assumptions

  - Always assume f, g, and h are continuous
  - Usually assume f, g, and h are C^1
  - Often assume f, g, and h are C^3

One-D Unconstrained Minimization: Newton's Method

$$\min_{x \in \mathbb{R}} \quad f(x),$$

- Assume $f(x)$ is C^2 functions $f(x)$
  - At a point a, the quadratic polynomial, $p(x)$

    $$p(x) \equiv f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

    is the second-order approximation of $f(x)$ at a
  - Approximately minimize f by minimizing $p(x)$
  - If $f''(a) > 0$, then p is convex, and $x_m = a - f'(a)/f''(a)$.
  - Hope: x_m is closer than a to the minimum.

- Newton's method:

  Algorithm 4.2 Newton's Method in $\mathbb{R}^1$

  Initialize. Choose initial guess x_0 and stopping parameters $\delta, \epsilon > 0$.

  Step 1. $x_{k+1} = x_k - f'(x_k)/f''(x_k)$.

  Step 2. If $|x_k - x_{k+1}| < \epsilon(1 + |x_k|)$ and $|f'(x_k)| < \delta$,
  STOP and report success; else go to step 1.

- Properties:
  - Newton's method finds critical points, that is, solutions to $f'(x) = 0$, not min or max.
  - If x_n converges to x^*, must check $f''(x^*)$ to check if min or max
  - Only find local extrema.
- Good news: convergence is locally quadratic.

Theorem 1 *Suppose that $f(x)$ is minimized at x^*, C^3 in a neighborhood of x^*, and that $f''(x^*) \neq 0$. Then there is some $\epsilon > 0$ such that if $|x_0 - x^*| < \epsilon$, then the x_n sequence defined in (4.1.2) converges quadratically to x^*; in particular,*

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^2} = \frac{1}{2}\left|\frac{f'''(x^*)}{f''(x^*)}\right| \tag{4.1.3}$$

is the quadratic rate of convergence.

- Consumer problem example:
  - Consumer has \$1; price of x is \$2, price of y is \$3, utility function is $x^{1/2}+2y^{1/2}$.
  - If θ is amount spent on x then we have

$$\max_{\theta} \quad \left(\frac{\theta}{2}\right)^{1/2} + 2\left(\frac{1-\theta}{3}\right)^{1/2} \tag{4.1.6}$$

  - Solution $\theta^* = 3/11 = .272727$
  - If $\theta_0 = 1/2$, Newton iteration is

$$0.5, 0.2595917942,\ 0.2724249335,\ 0.2727271048,\ 0.2727272727$$

    and magnitude of the errors are

$$2.3\,(-1)\,,\ 1.3\,(-2)\,,\ 3.1\,(-4)\,,\ 1.7\,(-7)\,,\ 4.8\,(-14)$$

- Problems with Newton's method
  - May not converge.
  - $f''(x)$ may be difficult to calculate.

Multidimensional Unconstrained Optimization: Comparison Methods

- Grid Search
  - Pick a finite set of points, X.
    - X could be a Cartesian grid:

$$V = \{v_i | i = 1, ..., n\}$$
$$X = \{x \in \mathbb{R}^n | \forall i, x_i \in V\}$$

    - X could be a low-discrepancy set of points (see Chapter 9)
  - Compute $f(x)$, $x \in X$, and locate max
  - Should always do some grid search first.

- Polytope Methods

Algorithm 4.3 Polytope Algorithm

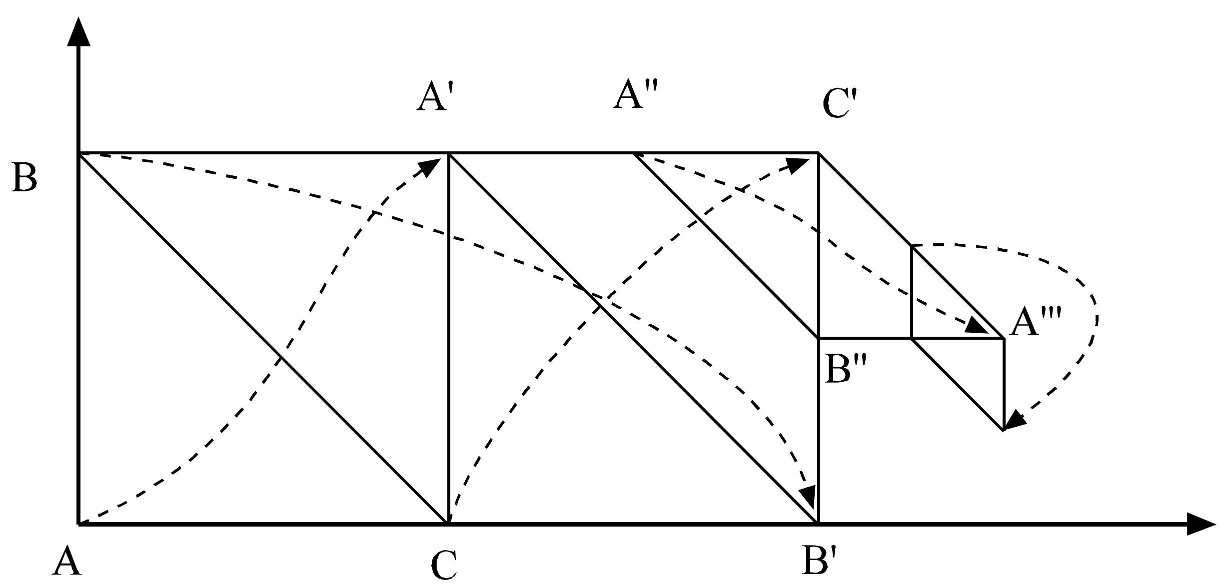
Initialize. Choose the stopping rule parameter ϵ. Choose an initial simplex $\{x^1, x^2, \cdots, x^{n+1}\}$.

Step 1. Reorder vertices so $f(x^i) \geq f(x^{i+1}),\ i = 1, \cdots, n$.

Step 2. Look for least i s.t. $f(x^i) > f(y^i)$ where y^i is reflection of x^i. If such an i exists, set $x^i = y^i$, and go to step 1. Otherwise, go to step 3.

Step 3. Stopping rule: If the width of the current simplex is less than ϵ, STOP. Otherwise, go to step 4.

Step 4. Shrink simplex: For $i = 1, 2, \cdots, n$ set $x^i = \frac{1}{2}(x^i + x^{n+1})$, and go to step 1.



Multidimensional Optimization: Newton's Method

- Idea: Given x^k, compute local quadratic approximation, $p(x)$, of $f(x)$ around x^k, and let x^{k+1} be max of $p(x)$

Algorithm 4.4 Newton's Method in $\mathbb{R}^n$

Initialize.	Choose x^0 and stopping parameters δ and $\epsilon > 0$.
Step 1.	Compute Hessian, $H(x^k)$, and gradient, $\bigtriangledown f(x^k)$, and solve $H(x^k)s^k = -(\bigtriangledown f(x^k))^\top$ for the step s^k.
Step 2.	$x^{k+1} = x^k + s^k$.
Step 3.	If $\parallel x^k - x^{k+1} \parallel < \epsilon(1 + \parallel x^k \parallel)$, go to step 4; else go to step 1.
Step 4.	If $\parallel \bigtriangledown f(x^{k+1}) \parallel < \delta(1 + \lvert f(x^{k+1}) \rvert)$, STOP and report success else STOP and report convergence to nonoptimal point.

- Stopping rule: Choose ε and δ to be bigger than square root of machine epsilon.

Theorem 2 *Suppose that $f(x)$ is C^3, minimized at x^*, and that $H(x^*)$ is nonsingular. Then there is some $\epsilon > 0$ such that if $\| x^0 - x^* \| < \epsilon$, then the sequence defined in (4.3.1) converges quadratically to x^*.*

- Problems with Newton's method:
  - May not converge
  - Computational demands may be excessive
    - need at least $\mathcal{O}(n^2)$ time to compute $H(x^k)$, perhaps more if one does not have efficient code for $H(x)$
    - need $\mathcal{O}(n^2)$ space for $H(x^k)$
    - need $\mathcal{O}(n^3)$ time to solve $H(x^k)s^k = -(\nabla f(x^k))^\top$ for s^k
  - May converge to local solution, not global solution
  - We now consider methods which solve these problems.

Direction Set Methods

- Problem: may not converge, or go to wrong kind of extremum
- Solution: if we always move uphill, we will eventually get to a local maximum

Algorithm 4.5 Generic Direction Method

Initialize.	Choose initial x^0 and stopping parameters δ and $\epsilon > 0$.
Step 1.	Compute a search direction s^k.
Step 2.	Solve $\lambda_k = \arg\min_\lambda f(x^k + \lambda s^k)$.
Step 3.	$x^{k+1} = x^k + \lambda_k s^k$.
Step 4.	If $\parallel x^k - x^{k+1} \parallel < \epsilon(1 + \parallel x^k \parallel)$, go to step 5; else go to step 1.
Step 5.	If $\parallel \bigtriangledown f(x^{k+1}) \parallel < \delta(1 + f(x^{k+1}))$, STOP and report success; else STOP and report convergence to nonoptimal point.

- Possible direction set methods
  - Coordinate Directions
    - Let search directions be coordinate, x_1, x_2, etc.
    - Search direction $s_{2n+k} = x_k$
  - Steepest Descent: $s_k = \bigtriangledown f(x^k)$
  - Newton's Method with Line Search: $H_k s^k = -(\bigtriangledown f(x^k))^\top$
- These will always converge to a local optimum.

Quasi-Newton Methods

- Problem: Hessians are expensive to compute
- Solution: Don't need true Hessians (see Carter, 1993), so approximate them

Generic Quasi-Newton Method

Initialize.	Choose initial x^0, Hessian H^0 $(I)$and stopping parameters δ and $\epsilon > 0$.
Step 1.	Solve $H_k s^k = -(\bigtriangledown f(x^k))^\top$ for the search direction s^k.
Step 2.	Solve $\lambda_k = \arg\min_\lambda f(x^k + \lambda s^k)$
Step 3.	$x^{k+1} = x^k + \lambda_k s^k$.
Step 4.	Compute H_{k+1} using H_k, $\bigtriangledown f(x^{k+1})$, x^{k+1}, $\bigtriangledown f(x^k)$, etc.
Step 5.	If $\parallel x^k - x^{k+1} \parallel < \epsilon(1+ \parallel x^k \parallel)$, go to step 6;. else go to step 1
Step 6.	If $\parallel \bigtriangledown f(x^{k+1}) \parallel < \delta|1 + f(x^{k+1})|$, STOP and report success; else STOP and report convergence to nonoptimal point.

- Example: BFGS:

$$
\begin{aligned}
z_k &= x^{k+1} - x^k \\
y_k &= (\nabla f(x^{k+1}))^\top - (\nabla f(x^k))^\top \\
H_{k+1} &= H_k - \frac{H_k z_k z_k^\top H_k}{z_k^\top H_k z_k} + \frac{y_k y_k^\top}{y_k^\top z_k}
\end{aligned}
$$

  - Preserves positive definiteness
  - Uses only gradients that are already needed
  - Warning: denominators may get too small; should keep them away from zero since small z_k does not necessarily stop iteration.

- Note: The Hessian iterates H_k may not converge to true Hessian at solution, even if x_k converges to solution.

Monopoly Example

- We look at a simple monopoly pricing example:
  - Utility function: if M is spending on other goods,

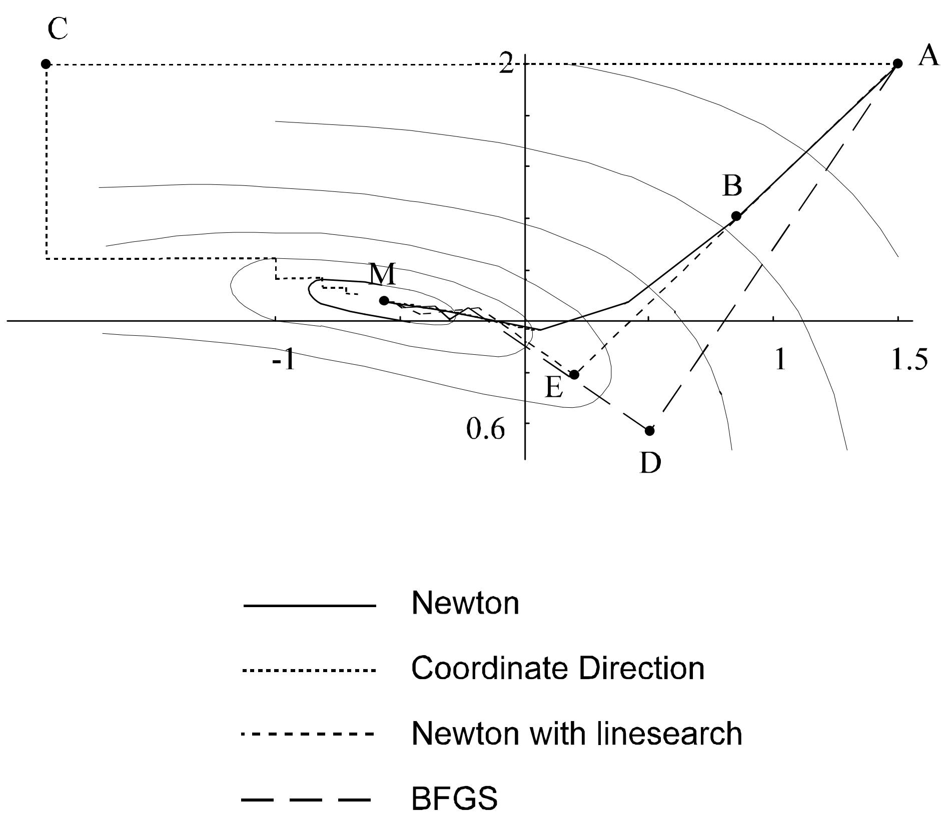
$$U(Y,Z) = (Y^\alpha + Z^\alpha)^{\eta/\alpha} + M = u(Y,Z) + M,$$

  - Output Y and Z implies prices of u_Y and u_Z.
  - Monopoly problem is

$$\max_{Y,Z} \Pi(Y,Z) \equiv Y u_Y(Y,Z) + Z u_Z(Y,Z) - C_Y(Y) - C_Z(Z), \quad (1)$$

  - Restate in terms of $y \equiv \ln Y$ and $z \equiv \ln Z$, $\pi(y,z) \equiv \Pi\left(e^y, e^z\right)$

$$\max_{y,z} \pi(y,z), \quad (2)$$



Conjugate Directions

- Problem: Hessians are expensive to compute
- Solution: Don't compute Hessians
- Neat trick in conjugate gradient method: If objective is quadratic, then we can avoid Hessian calculation

Algorithm 4.7 Conjugate Gradient Method

Initialize. Choose x^0, $s^0 = -\bigtriangledown f(x^0)$ and stopping parameters δ and $\epsilon > 0$.

Step 1. Solve $\lambda_k = \arg\min_\lambda f(x^k + \lambda s^k)$.

Step 2. Set $x^{k+1} = x^k + \lambda_k s^k$.

Step 3. Compute the search direction

$$s^{k+1} = -(\bigtriangledown f(x^{k+1})^\top) + \frac{\|\bigtriangledown f(x^{k+1})\|^2}{\|\bigtriangledown f(x^k)\|^2} s^k. \qquad (4.4.7)$$

Step 4. If $\| x^k - x^{k+1} \| > \epsilon(1+ \| x^k \|)$, go to step 1.
Otherwise, if $\| \bigtriangledown f(x^k) \| < \delta(1 + f(x^k))$,
STOP and report success;
else STOP and report convergence
to nonoptimal point.

- Works well for
  - large problems
  - problems with convex objective functions.

Example: A Dynamic Optimization Problem

- Life-cycle savings problem.
  - An individual lives for T periods
  - earns wages w_t in period $t, t = 1, \cdots, T$
  - consumes c_t in period t
  - earns interest on savings per period at rate r
  - utility function $\sum_{t=1}^{T} \beta^t u(c_t)$.
- Define S_t to be end-of-period savings:

$$S_{t+1} = (1+r)S_t + w_{t+1} - c_{t+1}.$$

  - The constraint $S_T = 0 = S_0$
  - Substitute $c_t = S_{t-1}(1+r) + w_t - S_t$
- Problem now has $T-1$ choices:

$$\begin{aligned} &\max_{S_t} \sum_{t=1}^{T} \beta^t u(S_{t-1}(1+r) + w_t - S_t) \\ &s.t. \quad S_T = S_0 = 0 \end{aligned} \tag{3}$$

  - Appears intractable for large T.
  - However, there are two ways to exploit the special structure of this problem and to efficiently solve this problem.

- Conjugate gradient method
  - Needs only gradients, which are easily computed here.
  - Objective is concave.
  - Example: $u(c) = -e^{-c}, \beta = 0.9, T = 6, r = 0.2,$
  - Initial condition $S_i = 0$.

Table 4.13

Conjugate gradient method for (3)

Iterate	S_1	S_2	S_3	S_4	S_5
1	0.4821	1.0170	1.015	1.326	0.7515
2	0.4423	0.7916	1.135	1.406	0.7270
3	0.3578	0.7631	1.120	1.446	0.7867
4	0.3680	0.7296	1.106	1.465	0.8258
5	0.3672	0.7351	1.100	1.468	0.8318
6	0.3710	0.7369	1.103	1.468	0.8370
7	0.3721	0.7447	1.110	1.476	0.8418
8	0.3742	0.7455	1.114	1.480	0.8427
9	0.3739	0.7456	1.114	1.480	0.8425

  - CG performs well
    * Converged at the 3-digit level after 7 iterations, 4 digits after 9
    * If objective were quadratic, convergence after 5 iterations.
    * Objective in (3) is not quadratic, but CG converges quickly.
    * Early iterates are surprisingly good approximations.

- Newton's method

  - Looks impractical if T large.
  - Hessian is tridiagonal (a sparse matrix), so Newton step is easy to compute.
  - Sparse Hessians are common in dynamic problems
  - *You* must recognize this and implement Newton or quasi-Newton method with sparse Hessians

Nonlinear Least Squares

- Objective function has form, $f^i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., m.$:

$$\min_x \frac{1}{2} \sum_{i=1}^{m} f^i(x)^2 \equiv S(x),$$

- Idea: use simple approximation of Hessian
- In econometric applications
  - $f^i(x)$ are $g(\beta, y^i)$,
    - $x = \beta$ is parameter vector
    - y^i are the data.
    - $g(\beta, y^i)$ is residual for observation i
  - $S(\beta)$ is the sum of squared residuals at β.
- Let $f(x)$ denote the column vector $(f^i(x))_{i=1}^m$.
  - Let $J(x)$ be the Jacobian of $f(x) \equiv (f^1(x), \ldots, f^m(x))^\top$.
  - Let $f_\ell^i \equiv \frac{\partial f^i}{\partial x_\ell}$ and $f_{j\ell}^i \equiv \frac{\partial^2 f^i}{\partial x_j \partial x_\ell}$.
  - The gradient of $S(x)$ is $J(x)^\top f$: $S_\ell(x) = \sum_{i=1}^m f_\ell^i(x) f^i(x)$.
  - The Hessian of $S(x)$ is $J(x)^\top J(x) + G(x)$, where

$$G_{j\ell}(x) = \sum_{i=1}^{m} f_{j\ell}^i(x) f^i(x).$$

- Special structure of the gradient and Hessian.
  - $f_j^i(x)$ terms are needed to compute gradient of $S(x)$.
  - If $f(x) = 0$, then Hessian is just $J(x)^\top J(x)$: easy to compute.
  - A problem where $f(x)$ is small at the solution is called a *small residual problem*; otherwise, it is a *large residual problem*.
- Gauss-Newton algorithm
  - Do Newton except use $J(x)^\top J(x)$ for Hessian approx.

$$s^k = -(J(x^k)^\top J(x^k))^{-1}(\nabla f(x^k))^\top \tag{4.5.1}$$

    and avoid computing second derivatives of f.
  - Natural to use for small residual problems.
  - Works very well when it works.

- Problems.
  - $J(x)^\top J(x)$ is likely to be poorly conditioned, since it is the "square" of a matrix.
  - $J(x)$ may be poorly conditioned itself, particularly in statistical contexts.
  - Gauss-Newton step may not be a descent direction.
- Solution: Levenberg-Marquardt algorithm.
  - Use $J(x)^\top J(x) + \lambda I$ for some scalar λ (I is identity matrix):

$$s^k = -(J(x^k)^\top J(x^k) + \lambda I)^{-1} (\nabla f(x^k))^\top$$

  - The λI term reduces conditioning problems.
  - s^k will be descent direction for large λ since s^k gets closer to steepest descent direction λ.

Linear Programming

- Canonical linear programming problem is

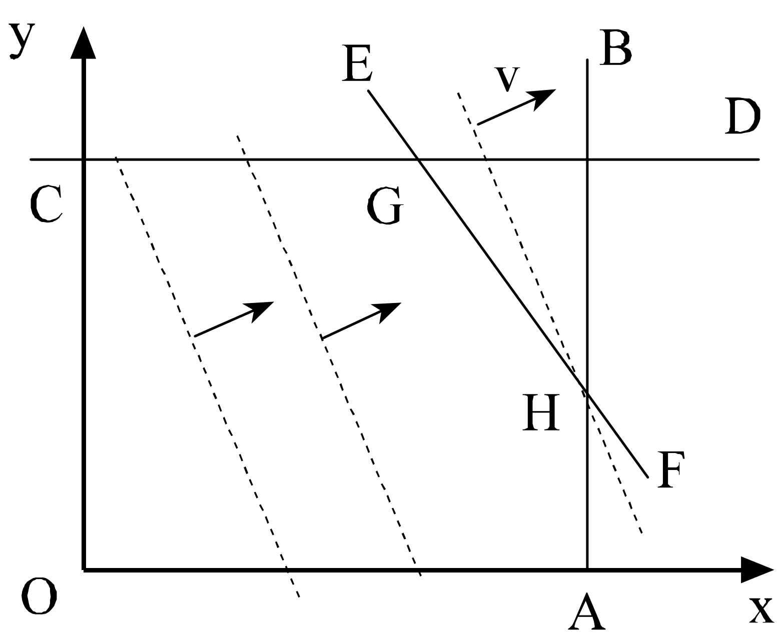
$$\begin{aligned} \min\nolimits_x a^\top x \\ s.t.\, Cx = b, \\ x \geq 0. \end{aligned} \tag{4}$$

  - $Dx \leq f$: use *slack variables*, s, and constraints $Dx + s = f, s \geq 0$.
  - $Dx \geq f$: use $Dx - s = f, s \geq 0$, s is vector of *surplus variables*.
  - $x \geq d$: define $y = x - d$ and min over y
  - x_i free: define $x_i = y_i - z_i$, add constraints $y_i, z_i \geq 0$, and min over (y_i, z_i).

- Basic method is the *simplex method*. Figure 4.4 shows example:

$$\begin{aligned} \min_{x,y} \; & -2x - y \\ s.t. \; & x + y \le 4, \quad x, y \ge 0, \\ & x \le 3, \quad y \le 2. \end{aligned}$$

  - Find some point on boundary of constraints, such as A.
  - Step 1: Note which constraints are active at A and points nearby.
  - Find feasible directions and choose steepest descent direction.
  - Figure 4.4 has two directions: from A: to B and to O, with B better.
  - Follow that direction to next vertex on constraint boundary (a linear equation), and go back to step 1.
  - Continue until no direction reduces the objective: point H.
  - Stops in finite time since there are only a finite set of vertices.



- General history
  - Goes back to Dantzig (1951).
  - Fast on average.
  - Worst case time is exponential in number of variables and constraints
  - Software implementations vary in numerical stability
- Interior point methods
  - Developed in 1980's
  - Better on large problems

Constrained Nonlinear Optimization

- General problem:

$$\begin{aligned} &\min_x f(x) \\ s.t.\ &g(x) = 0 \\ &h(x) \leq 0 \end{aligned} \tag{4.7.1}$$

  - $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$: n choices
  - $g : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$: m equality constraints
  - $h : X \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$: ℓ inequality constraints
  - f, g, and h are C^2 on X

- Kuhn-Tucker theorem: if there is a local minimum at x^* and a constraint qualification holds, then there are multipliers $\lambda^* \in \mathbb{R}^m$ and $\mu^* \in \mathbb{R}^\ell$ such that x^* is a *stationary*, or *critical* point of $\mathcal{L}$, the *Lagrangian*,

$$\mathcal{L}(x, \lambda, \mu) = f(x) + \lambda^\top g(x) + \mu^\top h(x) \tag{4.7.2}$$

- First-order conditions, $\mathcal{L}_x(x^*, \lambda^*, \mu^*) = 0$, imply that (λ^*, μ^*, x^*) solves

$$\begin{aligned} f_x + \lambda^\top g_x + \mu^\top h_x &= 0 \\ \mu_i h^i(x) &= 0, \quad i = 1, \cdots, \ell \\ g(x) &= 0 \\ h(x) &\leq 0 \\ \mu &\leq 0 \end{aligned} \tag{4.7.3}$$

A Kuhn-Tucker Approach

- Idea: try all possible Kuhn-Tucker systems and pick best
  - Let $\mathcal{J}$ be the set $\{1, 2, \cdots, \ell\}$.
  - For a subset $\mathcal{P} \subset \mathcal{J}$, define the $\mathcal{P}$ problem, corresponding to a combination of binding and nonbinding inequality constraints

$$\begin{aligned} g(x) &= 0 \\ h^i(x) &= 0\,, \quad i \in \mathcal{P}, \\ \mu^i &= 0\,, \quad i \in \mathcal{J} - \mathcal{P}\ , \\ f_x + \lambda^\top g_x + \mu^\top h_x &= 0. \end{aligned} \tag{4.7.4}$$

  - Solve (or attempt to do so) each $\mathcal{P}$-problem
  - Choose the best solution among those $\mathcal{P}$-problems with solutions consistent with all constraints.
- We can do better in general.

Penalty Function Approach

- Most constrained optimization methods use a *penalty function* approach:
  - Replace constrained problem with related unconstrained problem.
  - Permit anything, but make it "painful" to violate constraints.
- Penalty function: for canonical problem

$$\begin{aligned} \min_x \;& f(x) \\ s.t. \;& g(x) = a, \\ & h(x) \le b. \end{aligned} \tag{4.7.5}$$

construct the penalty function problem

$$\min_x \; f(x) + \frac{1}{2} P \left(\sum_i \left(g^i(x) - a_i \right)^2 + \sum_j \left(\max\left[0, h^j(x) - b_j\right] \right)^2 \right) \tag{4.7.6}$$

where $P > 0$ is the penalty parameter.

  - Denote the penalized objective in (4.7.6) $F(x; P, a, b)$.
  - Include a and b as parameters of $F(x; P, a, b)$.
  - If P is "infinite," then (4.7.5) and (4.7.6) are identical.
  - Hopefully, for large P, their solutions will be close.

- Problem: for large P, the Hessian of F, F_{xx}, is ill-conditioned at x away from the solution.
- Solution: solve a sequence of problems.
  - Solve $\min_x \; F(x; P_1, a, b)$ with a small choice of P_1 to get x^1.
  - Then execute the iteration

$$x^{k+1} \in \arg\min_x \; F(x; P_{k+1}, a, b) \tag{4.7.7}$$

    where we use x^k as initial guess in iteration $k+1$, and $F_{xx}(x^k; P_{k+1}, a, b)$ as the initial Hessian guess (which is hopefully not too ill-conditioned)
- Shadow prices in (4.7.5) and (4.7.7):
  - Shadow price of a_i in (4.7.6) is $F_{a_i} = P(g^i(x) - a_i)$.
  - Shadow price of b_j in (4.7.6) is F_{b_j}; $P(h^j(x) - b_j)$ if binding, 0 otherwise.
- Theorem: Penalty method works with convergence of x and shadow prices as P_k diverges (under mild conditions)
- Method works locally not necessarily globally
  - Consider $\min_{0 \leq x \leq 1} x^3$.
  - Penalty function is $x^3 + P[(\min[x, 0])^2 + \max[x - 1, 0])^2]$
  - Unbounded as $x \to -\infty$, but has local min at $x = 0$.

- Simple example
  - Consumer buys good y (price is 1) and good z (price is 2) with income 5.
  - Utility is $u(y,z) = \sqrt{yz}$.
  - Optimal consumption problem is

$$\begin{array}{l} \max_{y,z} \sqrt{yz} \\ s.t. \;\; y + 2z \leq 5. \end{array} \tag{4.7.8}$$

    with solution $(y^*, z^*) = (5/2, 5/4)$, $\lambda^* = 8^{-1/2}$.
  - Penalty function is

$$u(y,z) - \frac{1}{2}P(\max[0, y + 2z - 5])^2$$

  - Iterates are in Table 4.7 (stagnation due to finite precision)

Table 4.7

Penalty function method applied to (4.7.8)

k	P_k	$(y,z) - (y^*, z^*)$	Constraint violation	λ error
0	10	(8.8(-3), .015)	1.0(−1)	−5.9(−3)
1	10^2	(8.8(−4), 1.5(−3))	1.0(−2)	−5.5(−4)
2	10^3	(5.5(−5), 1.7(−4))	1.0(−3)	2.1(−2)
3	10^4	(−2.5(−4), 1.7(−4))	1.0(−4)	1.7(−4)
4	10^5	(−2.8(−4), 1.7(−4))	1.0(−5)	2.3(−4)

Barrier Function Approach

- Some methods use a *barrier function* approach:
  - Make it "painful" to nearly violate constraints.
  - Take unconstrained approach even though new objective is not defined everywhere
- For canonical problem

$$\begin{aligned} &\min_x f(x) \\ &s.t.\ h(x) \le b \end{aligned} \tag{4.7.5}$$

construct the barrier function problem

$$\min_x \ f(x) + \varepsilon \left(\sum_j G\left(h^j(x) - b_j\right) \right) \tag{4.7.6}$$

where $G(z) > 0$ whenever $z < b$, and infinite when $z = 0$.

- Possible barrier functions are

$$\begin{aligned} G(z) &= -\log(-z) \\ G(z) &= -1/z \end{aligned}$$

- Solve sequence of problems with $\varepsilon \to 0$

General applicability

- Penalty and barrier methods replace true constrained problem with a smooth, unconstrained problem
- True problem often has a boundary solution whereas penalty or barrier function has interior solution
- The basic penalty and barrier methods can be used more generally to transform economic problems with difficult boundary problems or singularities into more manageable problems with interior, regular solutions.

Domain Problems

- Constrained problems

$$\begin{aligned} &\min_x f(x) \\ &s.t.\ g(x) = 0 \\ &\qquad h(x) \le 0 \end{aligned} \tag{4.7.1}$$

  where $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, $g : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $h : X \subseteq \mathbb{R}^n \to \mathbb{R}^\ell$

- The penalty function approach produces an unconstrained problem,

$$\max_{x \in \mathbb{R}^n} F(x; P, a, b)$$

  where it is assumed that $F(x; P, a, b)$ is defined for all x.

- Example: Consumer demand problem

$$\begin{aligned} &\max_{y,z} u(y,z) \\ &s.t.\ p\ y + q\ z \le I. \end{aligned}$$

  – Penalty method

$$\max_{y,z}\ u(y,z) - \frac{1}{2} P(\max[0,\ p\ y + q\ z - I])^2$$

  – Problem: $u(y,z)$ will not be defined for all y and z, such as

$$\begin{aligned} u(y,z) &= \log y + \log z \\ u(y,z) &= y^{1/3} z^{1/4} \\ u(y,z) &= \left(y^{1/6} + z^{1/6}\right)^{7/2} \end{aligned}$$

  – Penalty method will crash when computer tries to evaluate $u(y,z)$!

- Solution
  - Strategy 1: Transform variables
    - If functions are defined only for $x_i > 0$, then reformulate in terms of $z_i = \log x_i$
    - For example, let $\widetilde{y} = \log y$, $\widetilde{z} = \log z$, and solve

$$\max_{\widetilde{y},\widetilde{z}} \; u(e^{\widetilde{y}}, e^{\widetilde{z}}) - \frac{1}{2}P(\max[0, \; p\, e^{\widetilde{y}} + q\, e^{\widetilde{z}} - I])^2$$

    - Problem: log transformation may not preserve shape; e.g., concave function of x may not be concave in $\log x$
  - Strategy 2: Alter objective and constraint functions
    - E.G.; replace $u(c) = \log c$ with, for some small $\varepsilon > 0$

$$\widetilde{u}(c) = \begin{cases} u(c), & c > \varepsilon \\ u(\varepsilon) + u'(\varepsilon)(c-\varepsilon) + u''(\varepsilon)(c-\varepsilon)^2/2, & c \leq \varepsilon \end{cases}$$

    - Maintains curvature
    - Equals real $u(c)$ on most of domain, which hopefully includes solution
    - Not as easy to apply to multivariate functions

Sequential Quadratic Method

- Special quadratic problem, linear constraints

$$\begin{array}{ll} \min_x & (x-a)^\top A\,(x-a) \\ s.t. & b\,(x-s)=0 \\ & c\,(x-q)\le 0 \end{array}$$

  has special methods

- Sequential Quadratic Method
  - Solution is stationary point of Lagrangian

$$\mathcal{L}(x,\lambda,\mu)=f(x)+\lambda^\top g(x)+\mu^\top h(x)$$

  - Suppose that the current guesses are (x^k, λ^k, μ^k).
  - Let step size s^{k+1} solve approximating quadratic problem

$$\begin{array}{ll} \min_s & \mathcal{L}_x(x^k,\lambda^k,\mu^k)(x^k-s)+(x^k-s)^\top\mathcal{L}_{xx}(x^k,\lambda^k,\mu^k)(x^k-s) \\ s.t. & g(x^k)+g_x(x^k)(x^k-s)=0 \\ & h(x^k)+h_x(x^k)(x^k-s)\le 0 \end{array}$$

  - The next iterate is $x^{k+1}=x^k+s^{k+1}$; λ and μ are also updated but we do not describe the detail here.
  - Proceed through a sequence of quadratic problems.
  - S.Q. method inherits many properties of Newton's method
    * local convergence
    * can use quasi-Newton and line search methods.

Active Set Approach

- Problems:
  - Kuhn-Tucker approach has too many combinations to check
    - some choices of $\mathcal{P}$ may have no solution
    - there may be multiple local solutions to others.
  - Penalty function methods are costly since all constraints are in (4.7.5), even if only a few bind at solution.
- Solution: refine K-T with a *good sequence* of subproblems.
  - Let $\mathcal{J}$ be the set $\{1, 2, \cdots, \ell\}$
  - for $\mathcal{P} \subset \mathcal{J}$, define the $\mathcal{P}$ problem

$$\begin{aligned} &\min_x f(x) \\ &s.t.\ \ g(x) = 0, \qquad\qquad (\mathcal{P}) \\ &\qquad h^i(x) \leq 0, \quad i \in \mathcal{P}. \end{aligned} \tag{4.7.10}$$

  - Choose an initial set of constraints, $\mathcal{P}$, and start to solve (4.7.10-$\mathcal{P}$).
  - Periodically drop constraints in $\mathcal{P}$ which fail to bind
  - Periodically add constraints which are violated.
  - Increase penalty parameters
- The simplex method for linear programing is really an active set method.

Efficient Outcomes with Adverse Selection

- Rothschild-Stiglitz-Wilson (RSW) model of insurance markets with adverse selection; we formulate it as an endowment problem
- All agents receive either e_1 or e_2, $e_1 > e_2$
  - type H: probability π^H of receiving e_1
  - type L : probability π^L of receiving e_1, $\pi^H > \pi^L$.
  - θ^H $(\theta^L = 1 - \theta^H)$ is fraction of type $H(L)$ agents.
  - Risks are independent across agents;
  - Infinite number of each type; invoke LLN.
- Social planner
  - offers insurance contracts; redistributes income across states and people
  - sees only individual's realized income, not his type
  - must break even.

- $y = (y_1, y_2)$ is net state-contingent total income
  - pays $e_1 - y_1$ to insurer and consumes y_1 if income is e_1
  - receives $y_2 - e_2$ and consumes y_2 otherwise.
- Type t expected utility with net income, y^t.

$$U^t(y^t) = \pi^t u^t(y_1^t) + (1 - \pi^t)u^t(y_2^t), \; t = H, L,$$

- Planner's profits are

$$\begin{aligned} \Pi(y^H, y^L) = & \; \theta^H(\pi^H(e_1 - y_1^H) + (1 - \pi^H)(e_2 - y_2^H)) \\ & + \theta^L(\pi^L(e_1 - y_1^L) + (1 - \pi^L)(e_2 - y_2^L)). \end{aligned} \tag{4.8.3}$$

- Social planner offers menu $\left(y^H, y^L\right)$ and lets agents choose
- $y^H, y^L \in \mathbb{R}^2$ *constrained efficient* if it solves

$$\begin{aligned} \max \; & \lambda U^H(y^H) + (1 - \lambda)U^L(y^L) \\ s.t. \; & U^H(y^H) \geq U^H(y^L), \\ & U^L(y^L) \geq U^L(y^H), \\ & \Pi(y^H, y^L) \geq 0, \end{aligned} \tag{4.8.4}$$

where $0 \leq \lambda \leq 1$ is the welfare weight of type H agents.

- Example (Rothschild-Stiglitz, Wilson, Miyazaki, Spence):
  - $e_1 = 1$, $e_2 = 0$, $\pi^H = 0.8$, $\lambda = 1$
  - $u(c) = -e^{-4c}$
  - $P_k = 10^{1+k/2}$; $P_k = 10^k$ did not work as well

(Modification of Table 4.9)

Adverse selection example

π^L	θ^H	(y_1^H, y_2^H)	(y_1^L, y_2^L)	IV: $U^L(y^L) - U^L(y^H)$	Profit
0.70	0.10	$0.87, 0.51$	$0.70, 0.70$	$-1(-10)$	$-1(-10)$
0.50	0.10	$0.92, 0.35$	$0.50, 0.50$	$-5(-14)$	$-1(-14)$
0.70	0.75	$0.82, 0.79$	$0.77, 0.77$	$-1(-12)$	$-6(-13)$
0.50	0.75	$0.797, 0.789$	$0.794, 0.794$	$-2(-12)$	$-6(-13)$

- The results do reflect the predictions of adverse selection ("hidden information") theory.
  - If θ^H small, there is no cross-subsidy. Type H agents receive actuarially fair contracts but must face risk to keep type L agents from pretending to be H.
  - If θ^H large, cross-subsidies arise: the numerous type H agents take actuarially unfair contracts but receive safer allocations.
  - Type L agents always receive a risk-free consumption since no one wants to pretend to be L.

Computing Nash Equilibrium

- A game with n players.
  - Player i: strategy set $S_i = \{s_{i1}, s_{i2}, \cdots, s_{iJ_i}\}$.
  - $S = \Pi_{i=1}^{n} S_i$ is set strategy combinations.
  - $M_i(q_i, \sigma_{-i})$ is payoff to i from mixed strategy q_i if others play σ_{-i}.
- Consider the function

$$v(\sigma) = \sum_{i=1}^{n} \sum_{s_{ij} \in S_i} \{\max [M_i(s_{ij}, \sigma_{-i}) - M_i(\sigma), 0]\}^2 .$$

Theorem 3 *(McKelvey) The solutions to*

$$\min_{\sigma} v(\sigma)$$
$$\sum \sigma^i (s_j) = 1$$
$$\sigma^i (s_j) \geq 0$$

are the Nash equilibria of (M, S) and they are also the zeros of $v(\sigma)$, and conversely.

- Tradeoffs
  - Reduces Nash computation to a minimization problem
  - There may be local optima where $v(\sigma) > 0$ and are not equilibria.

- Example: simple coordination game:

R

L		
	1, 1	0, 0
	0, 0	1, 1

  - p_j^i is prob. that player i plays his jth strategy.
  - Payoff for each player is $p_1^1 p_1^2 + p_2^1 p_2^2$.
  - Lyapunov function for this game is

$$v(p_1^1, p_2^1, p_1^2, p_2^2) = \sum_{i,j=1}^{2} \max[0, p_i^j - (p_1^1 p_1^2 + p_2^1 p_2^2)]^2.$$

  - Three global min (and three equilibria) are

$$\begin{aligned}(p_1^1,\ p_2^1,\ p_1^2,\ p_2^2) = &(1,\ 0,\ 1,\ 0),\\ &(0.5,\ 0.5,\ 0.5,\ 0.5),\\ &(0,\ 1,\ 0,\ 1)\end{aligned}$$

  - BFGS did well except it got hung up on saddle point, but such hangups are easily fixed.

Table 4.10: Coordination game

Iterate	(p_1^1, p_1^2)	(p_1^1, p_1^2)	(p_1^1, p_1^2)	(p_1^1, p_1^2)
0	(0.1, 0.25)	(0.9, 0.2)	(0.8, 0.95)	(0.25, 0.25)
1	(0.175, 0.100)	(0.45, 0.60)	(0.959, 8.96)	(0.25, 0.25)
2	(0.110, 0.082)	(0.471, 0.561)	(0.994, 0.961)	(0.25, 0.25)
3	(0, 0)	(0.485, 0.509)	(1.00, 1.00)	(0.25, 0.25)
4	(0, 0)	(0.496, 0.502)	(1.00, 1.00)	(0.25, 0.25)
5	(0, 0)	(0.500, 0.500)	(1.00, 1.00)	(0.25, 0.25)

State-of-the-art Algorithms

- None of these methods are best by themselves
- Modern algorithms use sophisticated combinations of
  - local approximation methods
  - penalty functions,
  - active set ideas
  - sequential quadratic approximation ideas.
- Software
  - Matlab optimization toolbox - basic implementations, but not best
  - Solnp: Yinyu Ye's alternative Matlab optimization toolbox - see http://dollar.biz.uiowa.edu/col/ye/matlab.html.
  - NPSOL, MINOS, SNOPT - best software; written in Fortran
    * NPSOL - very good for dense problems, good interface
    * MINOS - best for large problems with sparse constraint systems, clumsy interfacte
  - IMSL, NAG libraries - contains good algorithms (NPSOL is in NAG)
  - GAMS
    * excellent general package for optimization and nonlinear equation solving
    * includes MINOS with good interface
  - AMPL - good, user-friendly package
  - NEOS: a free resource where you can submit a problem and it is assigned to an idle computer in its network. Look at http://www-neos.mcs.anl.gov/.